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Analysis of a Non-Discriminating Criterion in Simple Additive Weighting Deep Hierarchy

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Abstract: In the current account, we present an analysis of a non-discriminating criterion under simple additive weighting synthesis, considering a deep decision hierarchy. A *non-discriminating criterion* describes a criterion where all decision alternatives under consideration perform equally. We question eliminating such a criterion from the decision hierarchy in search of simpler problem representation and computational efficiency. Yet, we prove such an approach may result in order misrepresentations between decision alternatives. This analysis is performed in the form of four research questions that relate to the detection of certain conditions under which such distortions in the order integrity of decision alternatives will occur, calculating the change in their final performances, distinguishing the alternatives whose performances are consistent, and examining the role of the normalization procedure adopted in averting such distortions when the non-discriminating criterion is ignored. Along these lines, this study provides clear inferences which are of interest to researchers and decision makers, using simple additive weighting and similar methods that rely on additive synthesis.

Keywords: multiple-criteria decision making; simple additive weighting; non-discriminating criterion; normalization; rank-order; rank reversal; independence axiom; decision hierarchy

MSC: 90B50; 91B06

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1. Introduction

In the current study, we analyzed an incident we stumbled upon a while ago through initially evaluating a multiple-criteria decision situation within the simple additive weighting (SAW) synthesis. This case relates to the status of a non-discriminating criterion residing in a deep SAW decision hierarchy. In this context, we called a criterion *non-discriminating* if the performances of decision alternatives according to this criterion are identical. In our implementation, a criterion of lower-level hierarchy turned out to be non-discriminating after the collection of empirical data, though that was found to be relevant to the situation of an ex-ante assessment of the decision-making problem. Whilst we considered removing this criterion from decision hierarchy, due to exploration of a minimal problem representation, subsequent calculations showed that such an approach results in distortion in the order integrity of choice set under evaluation. In fact, the weights for the remaining criteria at that locality were equal to each other, and hence this was not an expected outcome throughout the synthesis. In our view, if a decision alternative x is selected by the implementation of a particular decision-making algorithm, such as the SAW synthesis, and the performance of x is not any better than other alternatives under a criterion c, then x should again be selected by the same algorithm when criterion c is removed from the criteria set. According to this perspective, the incident on hand presents an opportunity that may suggest valuable inferences for decision makers with SAW method. For that reason, it is our aim in this account to track the roots of such a decision anomaly in the SAW synthesis and gather our findings from this analysis.

In the vast literature of multiple-criteria decision making, similar analyses of decision anomalies and inconsistencies are not rare. One immediate example corresponds to the well-known case of the *rank reversal* phenomenon. In that background, the enquiry into rank reversal and preservation is mainly concerned with behavior of the rank-order when new components are added to, or deleted from, the choice set of a decision-making problem. There exists significant early discussion, in particular, concerning the renowned analytic hierarchy process-AHP [1] considering the consequences under such inclusion–exclusion practices on the choice set and underlying decision hierarchy, e.g., [2,3]. Other relevant works investigated the meaning of "relative importance" [4], the essence of the "value function" [5] that assigns a real number to each alternative in the choice set, and implications of the independence axiom [6]. Consequently, Saaty's priority approach has been criticized due to its potential weakness towards rank reversals. In rejoinders [7,8], Saaty and colleagues aimed to point out that rank reversal is a natural outcome both with noxious and desirable influences; thus, preserving rank-order in all decision situations is not true in terms of both technical and cognitive frames. On the other hand, he extends the conventional method to a more general approach, namely the analytic network process [9], which accounts for inner- and inter-dependencies of the criteria and choice sets.

Similar to the line of research devoted to rank reversal, there exist plentiful papers that investigate decision-making anomalies and inconsistencies from the viewpoints of method validation, hierarchical decomposition, measurement scales, aggregation rules, criteria structuring and their importance, number comparison principles, and so on. As we did not aim to survey this body of work in an all-embracing manner in the current manuscript, we summarize our review of relevant literature in Table 1 for interested readers. In this itemized format, the *discussions, critiques, and investigations* column notes the aspect of a particular method or approach that is explored in the *source* paper; the *theories, methods, and illustrations* column displays which theory or method is utilized in such critical analysis, or indicates the choice for the illustration of main arguments of the source.

Table 1. Summary of enquiries into decision-making anomalies and inconsistencies.

Source	Discussions, Critiques, and Investigations	Theories, Methods, and Illustrations
[2]	Shows the case of rank reversal in the AHP method	Contrary example
[10]	Questions weak justification of the Eigenvector method	Geometric mean method
[11]	States that criteria importance is not independent of alternative performances	Correspondence condition
[12]	Criticizes validation of the AHP scale	Analysis of published examples
[6]	Notes that procedure of hierarchical decomposition leads to arbitrary rankings	Absolute measurement, rescaling
[13]	Shows that cost/benefit analysis with AHP method yields non-optimal solutions	Incremental analysis
[14]	Investigates anomalies in methods to calculate random indices	Experimental results
[3]	Illustrates rank reversal in the AHP method	Geometric mean method
[15]	Analyses the inverse inconsistency in the AHP method	Statistically significant random indices
[16]	Argues that an ordering is not actually existent in the Eigenvector method	Geometric estimation, linking pins, supermatrix
[17]	Proves rank reversal in the supermatrix approach	Multi-attribute value theory
[18]	Questions weak justification of the Eigenvector method	Left-right eigenvector asymmetry
[19]	Challenges the calculation of relative importance of criteria	Nomology, criteria structuring
[20]	Investigates top-down hierarchical process and aggregation rules	Proofs, illustrative examples
[21]	Questions main axioms of the AHP	Discussions
[22]	Suggests that Eigenvector method is not suitable to reconcile inconsistent input	Discussions
[23]	Discusses unsuitable cost/benefit priorities and ratios	Magnitude adjustment

Table 1. *Cont.*

Source	Discussions, Critiques, and Investigations	Theories, Methods, and Illustrations
[24]	Investigates violation of independence axiom	Analysis of published examples
[25]	Shows the case of rank reversal in the AHP method	Illustrative example
[26]	Discusses corncerns about scale misinterpretation and eigenvalue evaluation	Discussions, systematic literature review
[27]	Studies failure of consistency index in case of contradictory judgments	Illustrative examples
[28]	Analyses a non-discriminating criterion in the context of AHP method	Numerical example
[29]	Criticizes non-existence of an absolute zero in ratio scales	Preference function modelling
[30]	Discusses conditions for mathematical operations on measurement scales	Principle of reflection, homogenity
[31]	Experiments addition/deletion of a non-discriminating criterion in a hierarchy	Numerical examples
[32]	Challenges the additive synthesis	Commensurate priorities
[33]	Shows order violations of preference intensities under the fuzzy system	Proofs, fuzzy preference programming
[34]	Studies unjustified rank reversal between dis-preferred alternatives	Illustrative examples
[35]	Substitutes a polynomial-time procedure for analysis of extent in grey system	Probability theory, proofs
[36]	Exposes different rankings in upper- and lower-triangular interval judgments	Post-optimality analysis
[37]	Studies complexity reduction in the presence of ordinal data	Linear transformation
[38]	Notes the Simpson paradox in aggregation of ranked data	Non-parametric pair-wise procedures
[39]	Shows need for new measures of inconsistency in stochastic decision making	Kullback-Leibner divergence
[40]	Investigates non-discriminating criteria in the context of AHP method	Weight adjustment, dependence
[41]	Discusses the implication of an attractive but unattainable alternative	Experimental studies
[42]	Proves failure of a proposed alternative inconsistency index	Illustrations, contrary example
[43]	Treats the problem of incomplete information	Compensatory programming
[44]	Demonstrates rank reversal in TOPSIS method	Analysis of PIS and NIS, algorithm modification
[45]	Suggests alternative visual method to number comparison	Probability theory, visualization
[46]	Shows that there is a limit to increasing homogeneity by replicating preferences	Distance metrics, consensus measures
[47]	Studies rank reversal in the ANP method	Economic experiments
[48]	Argues that eigenvector priority function cause strong rank reversal	Numerical example
[49]	Provides a critical analysis of the Eigenvector method in group decision making	Proofs
[50]	Investigates rank reversal in VIKOR method empirically	Empirical evaluation
[51]	Comparative analysis of rank reversal in three methods	Case study in sustainable material selection
[52]	Investigates both the ordinal and multiplicative consistencies	Ordinal consistency index
[53]	Studies inferring strength-of-preference across individuals	Experiments with participants
[54]	Analyses rank reversal in data envelopment analysis	Empirical study
[55]	Investigates rank reversal	Method development-RAFSI
[56]	Investigates rank reversal	Method development-NRTOPSIS
[57]	Studies invariant reference points and scales	Theoretical analysis, computational examples
[58]	Explores robust rank preservation based on Gaussian distribution	Sensitivity analysis
[59]	Questions the scale constraints in distance-based decision making	Theoretical analysis, case studies
[60]	Studies correlations of violations in pre-defined conditions for final decisions	Laboratory experiment
[61]	Shows amplification of greyness degree in decision-making algorithms	Proofs, Monte-Carlo simulation

Not only related to the status of weakly-dominated decision alternatives, distortions in the order integrity may occur when other acts of a decision-making problem change. Finan and Hurley [25] named what we call a non-discriminating criterion the "wash" criterion, and similarly noted that the decision maker cannot differentiate the alternatives on that criterion; more simply, the linked decision alternatives have equal performances. On this basis, they questioned the deletion of a wash criterion from the criteria set of the AHP method in particular. In their implementation, they assumed a consistent decision matrix and investigated whether the conventional AHP synthesis conforms to elimination of wash criterion from hierarchies of single and multiple levels. They proved that elimination of the wash criterion does not affect the order integrity of decision alternatives in single-level hierarchies. Nevertheless, they also proved that this result does not extend to multiple-level hierarchies.

In a similar approach, Pérez et al. [31] discussed embedding a non-discriminating criterion into a multiple-level hierarchy to obtain the same result. They criticized the AHP method from the viewpoint that the rank-order does not respond to the introduction and deletion of the non-discriminating criterion in a coherent way, considering that in some situations the criteria set of a decision-making problem may not be defined beforehand, and analysis may be restricted to relevant criteria for which the performances of alternatives show meaningful disagreement through the implementation steps of the decision process.

Wijnmalen and Wedley [40] extended the previous result by Finan and Hurley [25] by relaxing the assumption of a consistent decision matrix. Their analysis of multiple-level hierarchies provided outcomes that are congruous to both Finan and Hurley [25] and Pérez et al. [31].

Turning to the case of our experience with the non-discriminating criterion in SAW synthesis, from a practical point of view, one may ponder that such criterion may be singled out from decision hierarchy due to incidences in simpler problem representation and computational efficiency. Nevertheless, as we prove in this work, SAW synthesis is exposed to possible distortions in the order integrity when such a criterion, in a lower level hierarchy, is instinctively removed from the criteria set. Therefore, our first research question in this analysis is structured as follows.

RQ1. In which circumstances is the SAW synthesis open to distortions in the order integrity when a non-discriminating criterion is ignored?

In our view, the arguments instituted through such an analysis should include convenient inferences for decision makers with the SAW method. On that account, we derive a condition to check the alleged order misrepresentation between any decision-alternative pair based on our findings. On the other hand, when such a misrepresentation is known to occur, the decision makers shall wonder by how much the final performance of each decision alternative will alter when ignoring the non-discriminating criterion. This leads to the formulation of a second research question.

RQ2. By how much will the final performance of each decision alternative change when a non-discriminating criterion is ignored?

Hence, based on this enquiry, we derive necessary arguments to measure change in the final performances of decision alternatives, with the exclusion of the non-discriminating criterion. In this situation, it is also a point of concern whether there will be any decision alternatives whose final performances are not mutable. This issue additionally leads to the following research question.

RQ3. Are there any decision alternatives whose final performances remain unchanged when a non-discriminating criterion is ignored?

Then, we show that when the non-discriminating criterion is removed, some alternative(s) may become *anchor(s)* whose performance does not alter overall results. We also analyze the role of normalization procedure by reflecting upon a substitute normalization scheme, based on the following premise.

RQ4. Will substitution in the normalization procedure guarantee conservation of the order integrity when a non-discriminating criterion is ignored?

We believe that our investigation into the above four research questions will yield a four-fold contribution to the narrow literature on the non-discriminating criterion, which we think has not yet been discussed with all practicalities as well as perils, especially given the fact that it has not yet been studied in the context of SAW synthesis.

To this end, this paper is organized as follows. In Section 2 we detail an analysis of the non-discriminating criterion considering a deep SAW hierarchy to generate arguments for the existence and detection of distortions in the order integrity. Section 3 is devoted to illustrating the properties implied by this analysis, as answers to the first three research questions, which may appeal to decision makers using the SAW method. In Section 4 we illustrate how to work with these constructs in an example multiple-criteria decision-making problem. Some remarks concerning our analysis and an exploration to the fourth research question are presented in Section 5. We finally summarize our conclusions from this study in Section 6.

2. Analysis

The SAW method is one of the earliest procedures used to assign values to elements of a choice set that is processed with regards to a criteria set. Its origin can be traced as far back as the foremost work of Churchman and Ackoff [62]. Since then, it has been practiced in numerous applications, both in technical and social contexts. A detailed account of the applications of the SAW method in a variety of decision-making situations can be found in a review by Abdullah and Adawiyah [63].

Briefly, the SAW method works as follows. Let x_i be n decision alternatives, indexed by i, that constitute the choice set, and c_j be m criteria, indexed by j that make up the criteria set. Denote the performances of alternatives over the criteria set by $a_{ij} \geq 0$ and associate a weight $w_j \geq 0$ with each criterion, such that:

$$\sum_{j=1}^{m} w_j = 1. \tag{1}$$

The criteria set is apportioned to two disjoint sets B and C, denoting the sets of benefit criteria (i.e., more denotes a better type) and cost criteria (i.e., less denotes a better type), respectively. Under this representation, the SAW method assumes a decision matrix $A = \left(a_{ij}\right)_{n\times m}$ to work through its normalized counterpart $\hat{A} = \left(\hat{a}_{ij}\right)_{n\times m}$ where

$$\hat{a}_{ij} = \begin{cases} \frac{a_{ij}}{a_{*j}} & \text{if} \quad j \in B \\ \frac{a_{*j}}{a_{ij}} & \text{if} \quad j \in C, \end{cases} \tag{2}$$

such that $a_{*j} = \max_i \left\{a_{ij}\right\}$ for $j \in B$, and $a_{*j} = \min_i \left\{a_{ij}\right\}$ for $j \in C$. Then, the overall performance a_i of alternative x_i is computed by using

$$a_i = \sum_{j=1}^{m} w_j \cdot \hat{a}_{ij}. \tag{3}$$

The greater the value of a_i the better the alternative x_i; hence, the rank-order is attained accordingly.

In the sequel, we analyze the status of a non-discriminating criterion, assuming the above SAW procedure and a deep hierarchy. For our purposes, consider the hierarchy illustrated in Figure 1. There exists a goal g at the first level, and a set of m criteria c_j for which weights $w_j \geq 0$ are assigned at the second level. Each criterion c_j is composed of $k(j)$ sub-criteria c_{jk}, with the exception of c_1, which, in addition, accommodates the non-discriminating criterion c_0 as a sub-criterion at the third level, as shown in this figure.

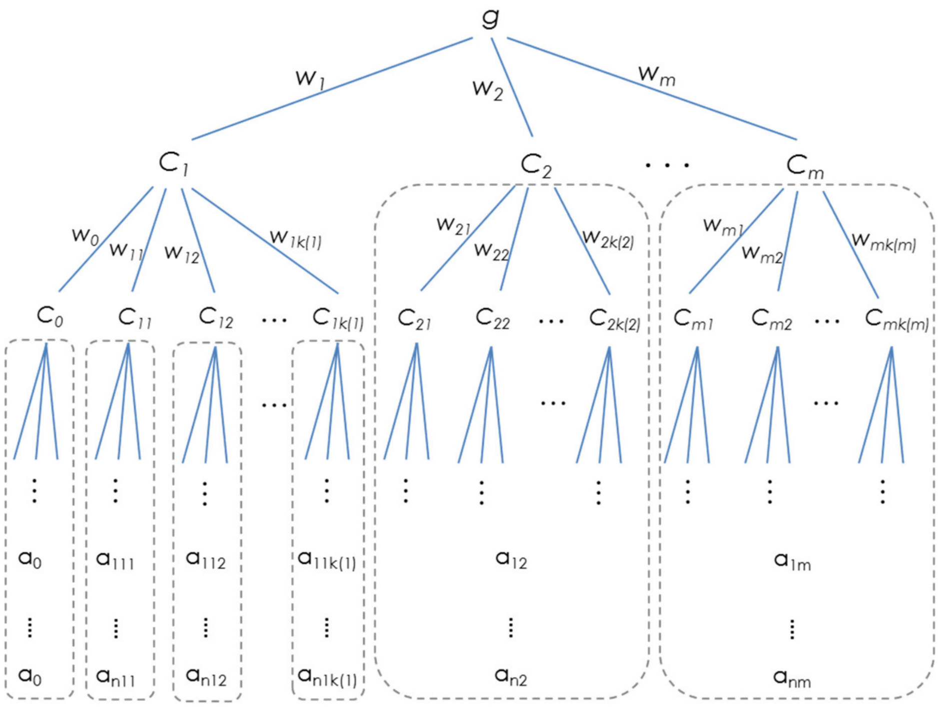

Figure 1. Deep SAW hierarchy.

Without loss of generality, any criterion that accommodates the non-discriminating criterion, and hence the sub-criteria of that criterion, can be re-indexed to conform to this setting. Moreover, there may be numerous other elements residing at lower levels of the hierarchy, which are not illustrated here; their existence is marked by three dots expanding below. The weights assigned to sub-criteria are then given by $w_{jk} \geq 0$, with the exception of the non-discriminating criterion, which assumes $w_0 \geq 0$. The weights attached to each criteria and sub-criteria satisfy normalization constraints; therefore, at the second level we have:

$$\sum_{j=1}^{m} w_j = 1, \tag{4}$$

and at the third level we have:

$$w_0 + \sum_{k=1}^{k(1)} w_{1k} = 1, \quad j = 1, \tag{5}$$

and

$$\sum_{k=1}^{k(j)} w_{jk} = 1, \quad \forall j,\, j \neq 1, \tag{6}$$

where similar construction is true for all levels below. We consider a choice set of n alternatives x_i and let a_{i1k} be the cumulative performances accumulated bottom-up by devouring the hierarchy under SAW synthesis up to the third level, and a_{ij} be those accumulated up to the second level. The extent of this operation is delimited with dashed contours in Figure 1. As the local calculations of performances below c_{1k} are not relevant to our analysis until the third level, and similarly those below c_2 to c_m are not relevant until the second level, we directly use cumulative performance results given by a_{i1k} and a_{ij}, respectively. The equal performances a_0 below the non-discriminating criterion, may occur in two cases. Either the non-discriminating criterion has no sub-criteria at the fourth level and the performances are simply a_0, or, in a less likely case, cumulative performances computed bottom-up for each alternative turn out to be equal at the third level, with the value a_0. We now present our result under this convention, where, for illustration purposes, the non-discriminating criterion is embedded in the third level. Yet, we note that similar arguments for any case where the non-discriminating criterion is embedded in a deeper level may be developed by following the same approach.

Theorem 1. *SAW synthesis is open to distortions in the order integrity upon removal of a non-discriminating criterion embedded in a deep hierarchy.*

Proof of Theorem 1. Suppose two cases where, for the first one, the non-discriminating criterion $\boldsymbol{c_0}$ is present, and for the latter one it is removed from the criteria set. We introduce a superscript 0 to denote the former case and break down the hierarchy, starting from $\boldsymbol{c_1}$. The decision matrices at this locality are in the form:

$$A_1^0 = \begin{pmatrix} a_0 & a_{111} & \cdots & a_{11k(1)} \\ a_0 & a_{211} & \cdots & a_{21k(1)} \\ \vdots & \vdots & a_{i1k} & \cdots \\ a_0 & a_{n11} & \cdots & a_{n1k(1)} \end{pmatrix}, A_1 = \begin{pmatrix} a_{111} & \cdots & a_{11k(1)} \\ a_{211} & \cdots & a_{21k(1)} \\ \vdots & a_{i1k} & \cdots \\ a_{n11} & \cdots & a_{n1k(1)} \end{pmatrix} \tag{7}$$

Suppose, without loss of generality, that all criteria in the hierarchy are benefit criteria and define $a_{*1k} = \max_i \{a_{i1k}\}$, $k = 1, \ldots, k(1)$. Then, normalized decision matrices are given by:

$$\hat{A}_1^0 = \begin{pmatrix} 1 & \frac{a_{111}}{a_{*11}} & \cdots & \frac{a_{11k(1)}}{a_{*1k(1)}} \\ 1 & \frac{a_{211}}{a_{*11}} & \cdots & \frac{a_{21k(1)}}{a_{*1k(1)}} \\ \vdots & \vdots & \frac{a_{i1k}}{a_{*1k}} & \vdots \\ 1 & \frac{a_{n11}}{a_{*11}} & \cdots & \frac{a_{n1k(1)}}{a_{*1k(1)}} \end{pmatrix}, \hat{A}_1 = \begin{pmatrix} \frac{a_{111}}{a_{*11}} & \cdots & \frac{a_{11k(1)}}{a_{*1k(1)}} \\ \frac{a_{211}}{a_{*11}} & \cdots & \frac{a_{21k(1)}}{a_{*1k(1)}} \\ \vdots & \frac{a_{i1k}}{a_{*1k}} & \vdots \\ \frac{a_{n11}}{a_{*11}} & \cdots & \frac{a_{n1k(1)}}{a_{*1k(1)}} \end{pmatrix} \tag{8}$$

Let

$$\hat{a}_{i1k} = \frac{a_{i1k}}{a_{*1k}}, \quad i = 1, \ldots n; k = 1, \ldots, k(1), \tag{9}$$

then, performances of each alternative under weights w_0, $w_{11}, \ldots, w_{1k(1)}$ for the case when non-discriminating criterion is included are given by:

$$\begin{array}{l} a_{11}^0 = w_0 + w_{11} \cdot \hat{a}_{111} + \ldots + w_{1k(1)} \cdot \hat{a}_{11k(1)}, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}^0 = w_0 + w_{11} \cdot \hat{a}_{n11} + \ldots + w_{1k(1)} \cdot \hat{a}_{n1k(1)}. \end{array} \tag{10}$$

On the other hand, when the non-discriminating criterion is removed, notice that normalization constraint for weights is distorted. However, we retrieve the totality:

$$\sum_{k=1}^{k(1)} w_{1k} = 1 - w_0 \tag{11}$$

to satisfy the normalization constraint with

$$w'_{1k} = \frac{w_{1k}}{1 - w_0}, \; k = 1, \ldots, k(1) \tag{12}$$

such that

$$\sum_{k=1}^{k(1)} w'_{1k} = 1. \tag{13}$$

Thus, the performance of each alternative under weights $w'_{11}, \ldots, w'_{1k(1)}$ for the case when the non-discriminating criterion is removed is given by:

$$\begin{array}{l} a_{11} = \frac{w_{11}}{1-w_0} \cdot \hat{a}_{11} + \ldots + \frac{w_{1k(1)}}{1-w_0} \cdot \hat{a}_{n1k(1)} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1} = \frac{w_{n11}}{1-w_0} \cdot \hat{a}_{n11} + \ldots + \frac{w_{1k(1)}}{1-w_0} \cdot \hat{a}_{n1k(1)} \end{array} \tag{14}$$

In general, we obtain:

$$a_{i1}^0 = w_0 + b_i , \quad i = 1, \ldots, n, \tag{15}$$

$$a_{i1} = \frac{1}{1-w_0} \cdot b_i, \ i = 1, \ldots, n, \tag{16}$$

where

$$b_i = \sum_{k=1}^{k(1)} w_{1k} \cdot \hat{a}_{i1k} , \quad i = 1, \ldots, n. \tag{17}$$

When it comes to processing the goal, decision matrices and their normalized counterparts are in the form:

$$A^0 = \begin{pmatrix} w_0 + b_1 & a_{12} & \cdots & a_{1m} \\ w_0 + b_2 & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & a_{ij} & \vdots \\ w_0 + b_n & a_{n2} & \cdots & a_{nm} \end{pmatrix}, A = \begin{pmatrix} \frac{1}{1-w_0} b_1 & a_{12} & \cdots & a_{1m} \\ \frac{1}{1-w_0} b_2 & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & a_{ij} & \vdots \\ \frac{1}{1-w_0} b_n & a_{n2} & \cdots & a_{nm} \end{pmatrix} \tag{18}$$

$$\hat{A}^0 = \begin{pmatrix} \frac{w_0+b_1}{w_0+b_*} & \frac{a_{12}}{a_{*2}} & \cdots & \frac{a_{1m}}{a_{*m}} \\ \frac{w_0+b_2}{w_0+b_*} & \frac{a_{22}}{a_{*2}} & \cdots & \frac{a_{2m}}{a_{*m}} \\ \vdots & \vdots & \frac{a_{ij}}{a_{*j}} & \vdots \\ \frac{w_0+b_n}{w_0+b_*} & \frac{a_{n2}}{a_{*2}} & \cdots & \frac{a_{nm}}{a_{*m}} \end{pmatrix}, \hat{A} = \begin{pmatrix} \frac{b_1}{b_*} & \frac{a_{12}}{a_{*2}} & \cdots & \frac{a_{1m}}{a_{*m}} \\ \frac{b_2}{b_*} & \frac{a_{22}}{a_{*2}} & \cdots & \frac{a_{2m}}{a_{*m}} \\ \vdots & \vdots & \frac{a_{ij}}{a_{*j}} & \vdots \\ \frac{b_n}{b_*} & \frac{a_{n2}}{a_{*2}} & \cdots & \frac{a_{nm}}{a_{*m}} \end{pmatrix} \tag{19}$$

where $b_* = \max_i \{b_i\}$ and $a_{*j} = \max_i \{a_{ij}\}$, $j = 2, \ldots, m$. Letting

$$\hat{a}_{ij} = \frac{a_{ij}}{a_{*j}} , \quad i = 1, \ldots, n; j = 2, \ldots, m, \tag{20}$$

and the performances of each alternative under weights $w_1, \ldots, w_m$ are derived as:

$$a_i^0 = w_1 \cdot \frac{w_0 + b_i}{w_0 + b_*} + u_i , \quad i = 1, \ldots, n, \tag{21}$$

$$a_i = w_1 \cdot \frac{b_i}{b_*} + u_i , \quad i = 1, \ldots, n, \tag{22}$$

where

$$u_i = \sum_{j=2}^{m} w_j \cdot \hat{a}_{ij}, \quad i = 1, \ldots, n. \tag{23}$$

To conclude this argument, consider two alternatives x_i and x_l. Let the difference between performances of x_i and x_l be d_{il} with $d_{il} = a_i - a_l$. When the non-discriminating criterion is included or excluded from the choice set, this difference is equal to:

$$d_{il}^0 = \frac{w_1}{w_0 + b_*} \cdot (b_i - b_l) + (u_i - u_l), \tag{24}$$

$$d_{il} = \frac{w_1}{b_*} \cdot (b_i - b_l) + (u_i - u_l), \tag{25}$$

respectively. In order to preserve the order integrity between x_i and x_l, it is required that the sign of this difference does not alter between two cases. Note that a change of sign is possible when the value of $(u_i - u_l)$, the second term of the function, ceases to compensate for the value of the first term due to re-scaling by erosion of w_0 at the denominator. As such, SAW synthesis is open to distortions in order integrity upon removal of a non-discriminating criterion. □

In the above, we have proved that SAW synthesis may suffer from inconsistencies in the final order, similar to the widely-known case with weakly-dominated alternatives added to and deleted from the choice set, but this time using a case in relation to a non-discriminating criterion which is included in, and excluded from, the criteria set. We stress that our aim was not to propose arguments that may be used to compute SAW output alternatively, nor do we claim that those are computationally efficient. Nonetheless, the above construction has immediate consequences in the form of some properties that may be resorted to by decision makers using the SAW method while working with such criteria sets.

3. Implications

In this section, we use arguments of the previous section in response to our first three research questions and generate properties that are of particular interest to decision makers using the SAW method.

One immediate result that does not require further clarification relates to circumstances in which the order integrity is distorted. Based on Theorem 1, the answer to *RQ1* is that such inconsistency occurs when the sign of differences between performances of two alternatives change. Consequently, we propose a condition of order preservation between two alternatives, as follows.

Theorem 2 (Condition of order preservation)**.** *Order between two alternatives x_i and x_l will be preserved upon removal of the non-discriminating criterion from a deep SAW hierarchy if*

$$sign\left(d_{il}^0\right) = sign(d_{il}). \tag{26}$$

Decision makers may be interested in finding the cardinality of change in the performance of any alternative when the non-discriminating criterion is removed from the criteria set. It may be useful to have such information beforehand, especially to promote fairness in delicate decision-making problems, such as budget allocation, academic ranking, distributing ancillary benefits, and those of similar nature. To that aim, the answer to *RQ2* is staged by the following theorem.

Theorem 3 (Cardinality of change at the performance of an alternative)**.** *The performance of an alternative x_i will alter by Δ_i when the non-discriminating criterion is removed from the criteria set where:*

$$\Delta_i = w_1 \cdot \left(\frac{b_i}{b_*} - \frac{w_0 + b_i}{w_0 + b_*} \right). \tag{27}$$

Proof of Theorem 3. If an existing non-discriminating criterion is removed, the performance of an alter native x_i will change by $\Delta_i = a_i - a_i^0$. Then, by (21)–(22), we have:

$$\Delta_i = a_i - a_i^0 = w_1 \cdot \frac{b_i}{b_*} + u_i - \left(w_1 \cdot \frac{w_0 + b_i}{w_0 + b_*} + u_i \right) = w_1 \cdot \left(\frac{b_i}{b_*} - \frac{w_0 + b_i}{w_0 + b_*} \right). \tag{28}$$

□

Decision makers may also want to know whether there will be any alternatives whose final performances will not change upon removal of the non-discriminating criterion. Such alternatives, based on their performance, are essential pivots that form the basis of redistribution of performances of other alternatives when the non-discriminating criterion is removed. We shall name such alternatives the *anchors.* Accordingly, the answer to *RQ3* is presented by the following theorem.

Theorem 4 (Existence and performances of anchors)**.** *There exists at least one anchor whose performance does not alter with removal of the non-discriminating criterion from the criteria set.*

Proof of Theorem 4. Consider the set $X^* = \{x_{i*} \,:\, b_{i*} = b_*\}$. It is easy to see that $X^* \neq \varnothing$, moreover, for an $x_{i*} \in X^*$ we have:

$$\Delta_{i*} = w_1 \cdot \left(\frac{b_*}{b_*} - \frac{w_0 + b_*}{w_0 + b_*} \right) = 0 \tag{29}$$

which shows $a_{i*} = a^0_{i*}$, then $x_{i*} \in X^*$ are called the *anchors*. □

4. Illustration of Theorems

In this section, we show how to practice with the theorems introduced in previous sections in an example where the arguments will be applied to analyze points raised by *RQ1*–*RQ3* under a criteria set including a non-discriminating criterion. For our purposes we introduce an example inspired by an original instance by Finan and Hurley [25] (p. 1029), illustrated in Figure 2 under a three-level hierarchy where the numbers below level three sub-criteria denote the performance of each alternative.

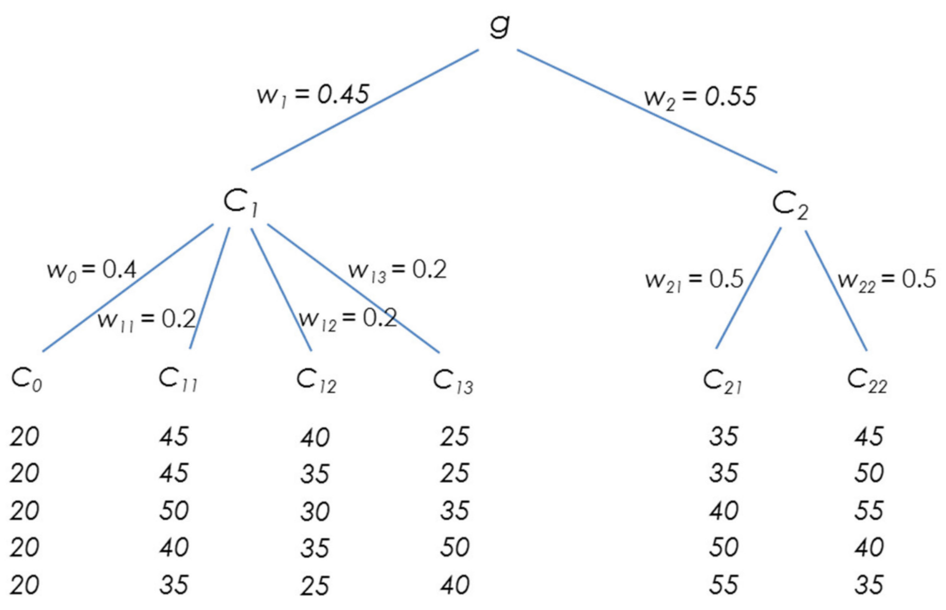

Figure 2. Illustration of Example 1.

Example 1. *Consider the choice set* $X = \{x_1, \ldots, x_5\}$ *and two main criteria* c_1 *and* c_2 *at the second level with weights* w_1 *and* w_2*. For* c_1 *there exist two sub-criteria,* c_0, c_{11}, c_{12}*, and* c_{13}*, with respective weights* w_0, w_{11}, w_{12}*, and* w_{13}*; for* c_2 *there are two sub-criteria,* c_{21} *and* c_{22}*, with weights* w_{21} *and* w_{22}*, respectively.*

Before solving the decision-making problem, suppose the decision maker is interested in:

(1) *investigating whether the order between* x_3 *and* x_4 *will be misrepresented upon the removal of* c_0*,*

(2) *finding out whether the performance of x_2 will increase or decrease, and to what extent, in the final results,*

(3) *detecting the alternative(s) whose performance will not be affected by such an exclusion.*

Based on our arguments, the application of Theorems 1–4 for completing tasks (1)–(3) is carried out as follows.

(1) To investigate an alleged order misrepresentation between x_3 and x_4, according to Theorem 1, we need $b_3 = 0.2 \cdot \left(\frac{50}{50} + \frac{30}{40} + \frac{35}{50}\right) = 0.49$, and, using similar calculation for b_4, we have $b_4 = 0.535$. On the other hand, below c_2 we obtain $a_{32} = 0.5 \cdot \left(\frac{40}{55} + \frac{55}{55}\right) = 0.8636$; again, using similar calculation for a_{42} gives $a_{42} = 0.8181$. Just for normalization purposes, we compute $a_{12} = 0.7272$, $a_{22} = 0.7727$, and $a_{52} = 0.8181$. Then, we obtain $\hat{a}_{32} = \frac{0.8636}{0.8636} = 1$ and $\hat{a}_{42} = \frac{0.8181}{0.8636} = 0.9473$, which help to compute $u_3 = 0.55 \cdot 1 = 0.55$ and $u_4 = 0.55 \cdot 0.9473 = 0.5210$. To use this information we also need b_*, which, upon resorting to the same procedure used to compute b_3 and b_4, and considering $b_1 = 0.48$, $b_2 = 0.455$, and $b_5 = 0.425$, we selected as $b_* = 0.535$. Therefore, we arrive at:

$$d_{34}^0 = \frac{0.45}{0.4 + 0.535} \cdot (0.49 - 0.535) + (0.55 - 0.521) = 0.0074$$

$$d_{34} = \frac{0.45}{0.535} \cdot (0.49 - 0.535) + (0.55 - 0.521) = -0.0088.$$

Using Theorem 2, we conclude that $sign\left(d_{34}^0\right) \neq sign(d_{34})$; hence, the order between x_3 and x_4 will be misrepresented if the non-discriminating criterion is excluded from the criteria set.

(2) For this particular purpose, we utilize Theorem 3 and obtain:

$$\Delta_2 = 0.45 \cdot \left(\frac{0.455}{0.535} - \frac{0.4 + 0.455}{0.4 + 0.535}\right) = 0.0288,$$

Which shows that the performance of x_2 will be reduced by 0.0288, subject to removal of the non-discriminating criterion.

(3) As we have $b_4 = b_* = 0.535$, according to Theorem 4, x_4 is the anchor alternative and its performance will not be affected by excluding the non-discriminating criterion.

For the purpose of assessing the above theorem results, we solved the decision-making problem given by this example with the SAW method when a non-discriminating criterion is included in the criteria set, and when it is removed from the criteria set. Related results are summarized in Appendix A. To check the above arguments, recall from Appendix A the overall rank-order $x_3 \succ x_4 \succ x_5 \succ x_2 \succ x_1$ when the non-discriminating criterion is included; however, when it is removed, we recall $x_4 \succ x_3 \succ x_5 \succ x_2 \succ x_1$, showing clearly the order misrepresentation between alternatives x_3 and x_4, as we have located using Theorem 2. Moreover, we also recall that the performance of x_2 reduces from $a_2^0 = 0.9036$ to $a_2 = 0.8748$, a difference of 0.0288, when the non-discriminating criterion is ignored, as we obtained by utilizing Theorem 3. Finally, the overall performance $a_4^0 = a_4 = 0.9710$ for x_4 shows that it is the anchor, whose performance is stable when the non-discriminating criterion is removed, as we located using Theorem 4.

5. Remarks

In this section we present three further remarks regarding our analysis in previous sections.

5.1. Non-Discriminating Criterion at Deeper Levels

Our analysis is irrespective of the level of a deep hierarchy in which the non-discriminating criterion is embedded. With the view that a single case is sufficient to prove probable non-integrity in the order under criteria sets with a non-discriminating criterion, and drawing on the SAW synthesis, as well as for a clear presentation of ideas, we considered a non-discriminating criterion embedded at the third level of a deep hierarchy. However, the procedure we detailed is adaptable, in the sense that one can generate similar arguments for cases where the non-discriminating criterion is embedded in deeper levels, if such analysis is anticipated.

To follow a process that is consistent with our steps one needs to ensure, if necessary with a re-indexing of the relevant criteria, that the upper level criteria for which the non-discriminating criterion is a sub-criterion all reside in the first branch of the hierarchy on the left-hand side that backtracks from the non-discriminating criterion to the goal, as shown in Figure 1. This establishes that all such criteria will be indexed by using ones only, where the necessary number of indices for each criteria is one less than its level. Then, keeping the deeper level overall performances, that are relevant to non-discriminating criterion, in the first column of the upper level decision matrix, and those that are irrelevant to non-discriminating criterion in other columns of this matrix, and continuing with this routine, one can simply generate the necessary arguments bottom-up. One also needs to note that they can be quite cumbersome, because, as the non-discriminating criterion is placed at deeper levels, additional parameters, similar to b_i and u_i, that keep track of the calculations both relevant and not relevant to the non-discriminating criterion need to be utilized.

5.2. Local Weights and Independence

When a non-discriminating criterion is removed, sub-criteria weights at its locality no longer add up to one to ensure unity. The new totality under effect is $1 - w_0$; the scale is once distorted and needs normalization. Local weights are normalized by adhering to this totality so that unity is restored, but, as we have reduced the process to the evaporation of w_0 at the denominator of d^0_{il} in Theorem 1, the integrity of the order is put at risk this time. Moreover, that is still the case even though all sub-criteria at that locality assume the same weights. We tried to illustrate this latter argument with equal prior weights $w_{11} = w_{12} = w_{13} = 0.2$ in Example 1, presented in the previous section.

The notion of *independence* in a decision hierarchy indicates that criteria weights, at some level, shall be determined independent of their association with lower-level constituents. It is a valuable construct that helps model a structured hierarchy and allow a step-by-step implementation of the tabbed method, by steering its comprehension. If this consideration is abandoned, one may tune-up local weights upon removal of the non-discriminating criterion by taking associations between hierarchy constituents into account, and then imposing preservation of the order integrity. However, such a "supermatrix" mode is not the case for SAW synthesis.

5.3. Choice of Normalization

SAW synthesis requires the utilization of linear normalization to ensure unity in calculations. In its original form, which is of particular interest to the analysis in previous sections, and without loss of generality under benefit criteria, SAW calls for linear normalizations of type:

$$(L1) \quad \hat{o}_{ij} = \frac{o_{ij}}{o_{*j}}\,, \quad j = 1, \ldots, m, \tag{30}$$

where o_{ij} are objects under normalization process and $o_{*j} = \max_i \{o_{ij}\}$, $j = 1, \ldots, m$.

There exists two other linear normalization methods whose cases may be examined in terms of whether a substitution in the normalization method contorts the main-order misrepresentation result in general. These normalization methods work as follows for benefit criteria:

$$(L2) \quad \hat{o}'_{ij} = \frac{o_{ij}}{\sum_{i=1}^{n} o_{ij}}\,, \quad j = 1, \ldots, m, \tag{31}$$

and

$$(L3) \quad \hat{o}''_{ij} = \frac{o_{ij} - o^-_j}{o_{*j} - o^-_j}\,, \quad j = 1, \ldots, m, \tag{32}$$

where $o_j^- = \min_i \{o_{ij}\}$, $j = 1, \ldots, m$.

Normalization with $L3$ is seldom preferred in SAW applications, and known to produce highly biased overall performances due to disfigurements in two aspects. First, it sets up the bottom-line at the minimal performance, resulting in unfair assessments with increasing severity as the range of performances (i.e., the denominator) gets smaller. Second, it assigns a normalized value of zero to the minimal performance under each criterion, which discredits the minimal performing alternative. We find it particularly unsuitable for our analysis, as $L3$ does not assume a non-discriminating criterion and would result in assignment of indefinite terms to its column in the normalized decision matrix immediate level above. For this reason, we single out $L3$ from further discussion.

On the other hand, when $L2$ is substituted for $L1$ in Example 1, in a surprising result, such substitution averted the order misrepresentation. We summarized the related calculations in Appendix B. Nevertheless, this instance does not contort our results in general, as it is a special case where the sum of elements in the c_1 column of the decision matrix utilized for the goal add up to 1; hence, its normalized counterpart is directly supplied with performances of the level below, with no denominators attached, and this, by chance, numerically helps to avoid order misrepresentation between x_3 and x_4. To see this, let us modify b_i parameters as follows:

$$b_i' = \sum_{k=1}^{k(1)} w_{1k} \cdot \frac{a_{i1k}}{\sum_{i=1}^{n} a_{i1k}}, \quad i = 1, \ldots, n, \tag{33}$$

where we append a prime to differentiate it from b_i utilized under $L1$ normalization. Then, the first column of the decision matrix A^0 for g is composed of the terms

$$\frac{w_0}{n} + b_i', \quad i = 1, \ldots, n, \tag{34}$$

whose sum is

$$w_0 + \sum_{i=1}^{n} b_i'. \tag{35}$$

Observe that, according to the above definition of b_i', for their sum we obtain

$$\sum_{i=1}^{n} b_i' = \sum_{k=1}^{k(1)} w_{1k} = 1 - w_0, \tag{36}$$

and that the totality (35) therefore adds up to 1. A similar case is also true when the non-discriminating criterion is removed; hence, the first column of the decision matrix A for g is composed this time of the terms:

$$\frac{1}{1 - w_0} \cdot b_i', \quad i = 1, \ldots, n, \tag{37}$$

whose sum is

$$\frac{1}{1 - w_0} \cdot \sum_{i=1}^{n} b_i', \tag{38}$$

which again adds up to 1.

As we have noted, this interesting instance does not extend to conclude that substituting $L2$ for $L1$ averts distortions in the order integrity in general. To illustrate this, and to answer our fourth research question, we now analyze another example illustrated in Figure 3, in which a non-discriminating criterion is embedded in the fourth level, and the substitution of $L2$ fails to avert the observed distortion in the order integrity.

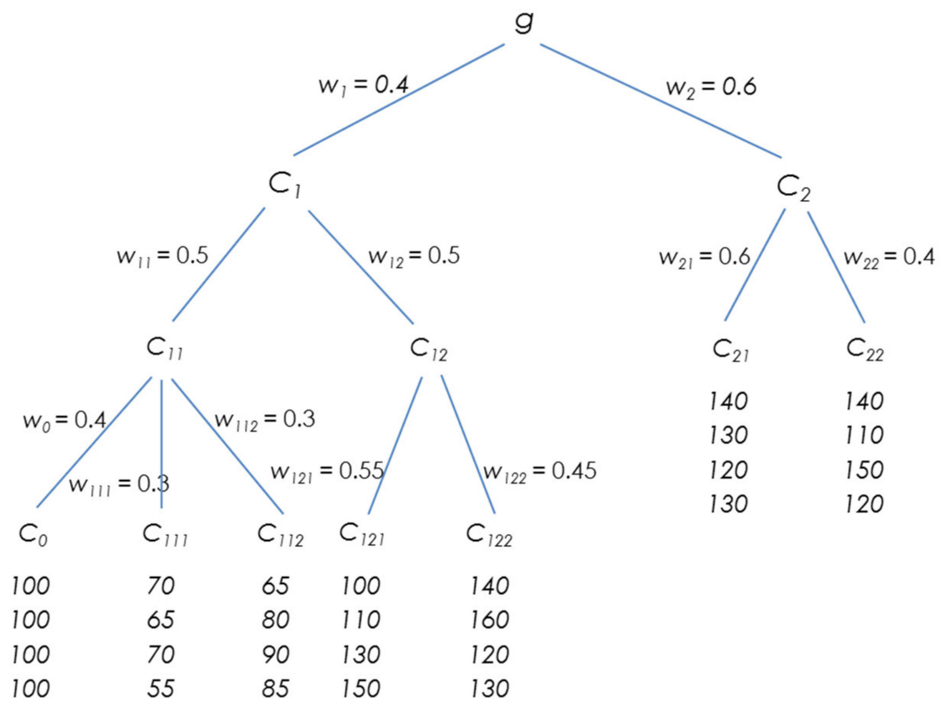

Figure 3. Illustration of Example 2.

Example 2. *Consider the choice set $X = \{x_1, \ldots, x_4\}$ and two main criteria c_1 and c_2 at the second level with weights w_1 and w_2. For c_1 there exists two sub-criteria, c_{11} and c_{12}, with weights w_{11} and w_{12}; for c_2 there are two sub-criteria, c_{21} and c_{22}, with weights w_{21} and w_{22} at the third level, respectively. At the fourth level, for c_{11}, there are three sub-criteria c_{111}, c_{112}, and the non-discriminating criterion c_0, with weights w_{111}, w_{112}, and w_0, respectively. Similarly, for c_{12} there are two sub-criteria, c_{121} and c_{122}, with weights w_{121} and w_{122} at the same level, respectively.*

A solution to the underlying decision-making problem with SAW synthesis is summarized in Appendix C. According to this solution, we obtain the rank-order $x_1 \succ x_3 \succ x_4 \succ x_2$ when the non-discriminating criterion is included. Nevertheless, when it is ignored we obtain the rank-order $x_3 \succ x_1 \succ x_4 \succ x_2$, showing a clear order misrepresentation between decision alternatives x_1 and x_3.

On the other hand, we substituted $L2$ as the normalization procedure and reiterated the SAW synthesis. This implementation is summarized in Appendix D. Note that the numerical values of final performances in this case come out closer than those obtained using $L1$, owing to repetitive normalizations over the sums of associated constituents. Nevertheless, we observe that substituting $L2$ did not help in averting the order misrepresentation between decision alternatives x_1 and x_3. Unfortunately, the answer to our fourth research question is not positive; hence, using a substitute normalization procedure do not guarantee conservation of the order integrity in a decision-making problem when the non-discriminating criterion is ignored.

6. Conclusions

In the current note we presented an analysis of the non-discriminating criterion under SAW synthesis and a deep decision hierarchy. Our analysis is transmuted in the form of four research questions that are of interest to researchers, decision makers, and practitioners using the SAW method.

We particularly questioned consequences with ignoring such a criterion, and found that doing so results in noxious distortions in the order integrity of the decision-making problem on hand. Based on this finding, we derived a condition to check order preservation—and naturally probable order misrepresentations as well—between decision alternatives when the non-discriminating criterion is ignored. We also provide arguments to compute the cardinality of change in the final performances of each decision alternative, in such a case. Moreover, we proved the existence of performances which do not alter in the final results, and named the associated decision alternatives as anchors. As a last resort, we explored the role of a normalization procedure adopted through SAW implementation in preventing order misrepresentations caused by ignoring the non-discriminating criterion. However, we found that substituting the normalization procedure does not help to overcome such misrepresentations,4 in general.

Finally, there is an important question in front of researchers, decision makers, and practitioners using the SAW method:

Should a non-discriminating criterion be excluded from the criteria set through SAW implementation?

In the domain of AHP, Liberatore and Nydick [28] noted that eliminating an essential sub-criterion affects the importance of the criterion of which the sub-criterion was a part. Thus, recognizing the essence of independence, they concluded that the criteria must be re-assessed when a non-discriminating criterion is excluded from the analysis; otherwise, when the re-assessment is not implemented, they stress that the non-discriminating criterion should not be eliminated. Similarly, Wijnmalen and Wedley's [40] analysis suggests including all criteria that are relevant to a decision-making situation, including non-discriminating criteria. Congruent results may be found in studies by Finan and Hurley [25] and Pérez et al. [31].

Our analysis using the SAW method produced implications that are in line with the above studies. When a non-discriminating criterion is excluded, it was evident that SAW potentially produces an altered rank-order and odd adjustments in overall performances, with some being constant. Apparently, it is not proper practice of the SAW method when one singles out a non-discriminating criterion from the analysis inattentively, given that it was found to be relevant to an ex-ante assessment of the decision situation and included in the criteria set beforehand.

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Appendix A

Table A1. Summary of Example 1 solution with SAW synthesis.

	Criteria								
Matrices	A_1^0				A_1			A_2	
	20	45	40	25	45	40	25	35	45
	20	45	35	25	45	35	25	35	50
	20	50	30	35	50	30	35	40	55
	20	40	35	50	40	35	50	50	40
	20	35	25	40	35	25	40	55	35
max	20	50	40	50	50	40	50	55	55
Normalized matrices	$\hat{A}_1^0$				$\hat{A}_1$			$\hat{A}_2$	
	1	0.9	1	0.5	0.9	1	0.5	0.6363	0.8181
	1	0.9	0.875	0.5	0.9	0.875	0.5	0.6363	0.9090
	1	1	0.75	0.7	1	0.75	0.7	0.7272	1
	1	0.8	0.875	1	0.8	0.875	1	0.9090	0.7272
	1	0.7	0.625	0.8	0.7	0.625	0.8	1	0.6363
weights	0.4	0.2	0.2	0.2	0.3333	0.3333	0.3333	0.5	0.5
performances	$a_{11}^0 =$	0.88			$a_{11} =$	0.7992		$a_{12} =$	0.7272
	$a_{21}^0 =$	0.855			$a_{21} =$	0.7575		$a_{22} =$	0.7727
	$a_{31}^0 =$	0.89			$a_{31} =$	0.8158		$a_{32} =$	0.8636
	$a_{41}^0 =$	0.935			$a_{41} =$	0.8907		$a_{42} =$	0.8181
	$a_{51}^0 =$	0.825			$a_{51} =$	0.7076		$a_{52} =$	0.8181
	goal								
matrices	A^0				A				
	0.88	0.7272			0.7992	0.7272			
	0.855	0.7727			0.7575	0.7727			
	0.89	0.8636			0.8158	0.8636			
	0.935	0.8181			0.8907	0.8181			
	0.825	0.8181			0.7076	0.8181			
max	0.935	0.8636			0.8907	0.8636			
Normalized matrices	$\hat{A}^0$				$\hat{A}$				
	0.9411	0.8421			0.8971	0.8421			
	0.9144	0.8947			0.8504	0.8947			
	0.9518	1			0.9158	1			
	1	0.9473			1	0.9473			
	0.8823	0.9473			0.7943	0.9473			
weights	0.45	0.55			0.45	0.55			
performances	$a_1^0 =$	0.8866			$a_1 =$	0.8668			
	$a_2^0 =$	0.9036			$a_2 =$	0.8748			
	$a_3^0 =$	0.9783			$a_3 =$	0.9621			
	$a_4^0 =$	0.9710			$a_4 =$	0.9710			
	$a_5^0 =$	0.9181			$a_5 =$	0.8785			

Appendix B

Table A2. Summary of Example 1 solution under $L2$ normalization.

					Criteria				
Matrices	A_1^0				A_1			A_2	
	20	45	40	25	45	40	25	35	45
	20	45	35	25	45	35	25	35	50
	20	50	30	35	50	30	35	40	55
	20	40	35	50	40	35	50	50	40
	20	35	25	40	35	25	40	55	35
sum	100	215	165	180	215	165	180	215	225
Normalized matrices	$\hat{A}_1^0$				$\hat{A}_1$			$\hat{A}_2$	
	0.2	0.2093	0.2424	0.1428	0.2093	0.2424	0.1428	0.1627	0.2
	0.2	0.2093	0.2121	0.1428	0.2093	0.2121	0.1428	0.1627	0.2222
	0.2	0.2325	0.1818	0.2	0.2325	0.1818	0.2	0.1860	0.2444
	0.2	0.1860	0.2121	0.2857	0.1860	0.2121	0.2857	0.2325	0.1777
	0.2	0.1627	0.1515	0.2285	0.1627	0.1515	0.2285	0.2558	0.1555
weights	0.4	0.2	0.2	0.2	0.3333	0.3333	0.3333	0.5	0.5
performances	$a_{11}^0 =$	0.1989			$a_{11} =$	0.1981		$a_{12} =$	0.1813
	$a_{21}^0 =$	0.1928			$a_{21} =$	0.1880		$a_{22} =$	0.1925
	$a_{31}^0 =$	0.2028			$a_{31} =$	0.2047		$a_{32} =$	0.2152
	$a_{41}^0 =$	0.2167			$a_{41} =$	0.2279		$a_{42} =$	0.2051
	$a_{51}^0 =$	0.1885			$a_{51} =$	0.1809		$a_{52} =$	0.2056
					goal				
matrices	A^0				A				
	0.1989	0.1813			0.1981	0.1813			
	0.1928	0.1925			0.1880	0.1925			
	0.2028	0.2152			0.2047	0.2152			
	0.2167	0.2051			0.2279	0.2051			
	0.1885	0.2056			0.1809	0.2056			
sum	1	1			1	1			
Normalized matrices	$\hat{A}^0$				$\hat{A}$				
	0.1989	0.1813			0.1981	0.1813			
	0.1928	0.1925			0.1880	0.1925			
	0.2028	0.2152			0.2047	0.2152			
	0.2167	0.2051			0.2279	0.2051			
	0.1885	0.2056			0.1809	0.2056			
weights	0.45	0.55			0.45	0.55			
performances	$a_1^0 =$	0.1892			$a_1 =$	0.1889			
	$a_2^0 =$	0.1926			$a_2 =$	0.1905			
	$a_3^0 =$	0.2096			$a_3 =$	0.2105			
	$a_4^0 =$	0.2103			$a_4 =$	0.2154			
	$a_5^0 =$	0.1979			$a_5 =$	0.1945			

Appendix C

Table A3. Summary of Example 2 solution with SAW synthesis.

				Criteria			
Matrices		A^0_{11}			A_{11}		A_{12}
	100	70	65	70	65	100	140
	100	65	80	65	80	110	160
	100	70	90	70	90	130	120
	100	55	85	55	85	150	130
max	100	70	90	70	90	150	160
normalized matrices		$\hat{A}^0_{11}$			$\hat{A}_{11}$		$\hat{A}_{12}$
	1	1	0.7222	1	0.7222	0.6666	0.875
	1	0.9285	0.8888	0.9285	0.8888	0.7333	1
	1	1	1	1	1	0.8666	0.75
	1	0.7857	0.9444	0.7857	0.9444	1	0.8125
weights	0.4	0.3	0.3	0.5	0.5	0.55	0.45
performances	$a^0_{111} =$	0.9166		$a_{111} =$	0.8611	$a_{112} =$	0.7604
	$a^0_{211} =$	0.9452		$a_{211} =$	0.9087	$a_{212} =$	0.8533
	$a^0_{311} =$	1		$a_{311} =$	1	$a_{312} =$	0.8141
	$a^0_{411} =$	0.9190		$a_{411} =$	0.8650	$a_{412} =$	0.9156
matrices		A^0_1			A_1		A_2
	0.9166	0.7604		0.8611	0.7604	140	140
	0.9452	0.8533		0.9087	0.8533	130	110
	1	0.8141		1	0.8141	120	150
	0.9190	0.9156		0.8650	0.9156	130	120
max	1	0.9156		1	0.9156	140	150
normalized matrices		$\hat{A}^0_1$			$\hat{A}_1$		$\hat{A}_2$
	0.9166	0.8304		0.8611	0.8304	1	0.9333
	0.9452	0.9319		0.9087	0.9319	0.9285	0.7333
	1	0.8891		1	0.8891	0.8571	1
	0.9190	1		0.8650	1	0.9285	0.8
weights	0.5	0.5		0.5	0.5	0.6	0.4
performances	$a^0_{11} =$	0.8735		$a_{11} =$	0.8458	$a_{12} =$	0.9733
	$a^0_{21} =$	0.9386		$a_{21} =$	0.9203	$a_{22} =$	0.8504
	$a^0_{31} =$	0.9445		$a_{31} =$	0.9445	$a_{32} =$	0.9142
	$a^0_{41} =$	0.9595		$a_{41} =$	0.9325	$a_{42} =$	0.8771
				goal			
matrices		A^0			A		
	0.8735	0.9733		0.8458	0.9733		
	0.9386	0.8504		0.9203	0.8504		
	0.9445	0.9142		0.9445	0.9142		
	0.9595	0.8771		0.9325	0.8771		
max	0.9595	0.9733		0.9445	0.9733		

Table A3. *Cont.*

	goal			
normalized matrices	$\hat{A}^0$		$\hat{A}$	
	0.9104	1	0.8954	1
	0.9781	0.8737	0.9743	0.8737
	0.9844	0.9393	1	0.9393
	1	0.9011	0.9872	0.9011
weights	0.4	0.6	0.4	0.6
performances	$a_1^0 =$	0.9641	$a_1 =$	0.9581
	$a_2^0 =$	0.9155	$a_2 =$	0.9139
	$a_3^0 =$	0.9573	$a_3 =$	0.9636
	$a_4^0 =$	0.9407	$a_4 =$	0.9355

Appendix D

Table A4. Summary of Example 2 solution under $L2$ normalization.

	Criteria						
Matrices	A_{11}^0			A_{11}		A_{12}	
	100	70	65	70	65	100	140
	100	65	80	65	80	110	160
	100	70	90	70	90	130	120
	100	55	85	55	85	150	130
sum	400	260	320	260	320	490	550
normalized matrices	$\hat{A}_{11}^0$			$\hat{A}_{11}$		$\hat{A}_{12}$	
	0.25	0.2692	0.2031	0.2692	0.2031	0.2040	0.2545
	0.25	0.25	0.25	0.25	0.25	0.2244	0.2909
	0.25	0.2692	0.2812	0.2692	0.2812	0.2653	0.2181
	0.25	0.2115	0.2656	0.2115	0.2656	0.3061	0.2363
weights	0.4	0.3	0.3	0.5	0.5	0.55	0.45
performances	$a_{111}^0 =$	0.2417		$a_{111} =$	0.2361	$a_{112} =$	0.2267
	$a_{211}^0 =$	0.25		$a_{211} =$	0.25	$a_{212} =$	0.2543
	$a_{311}^0 =$	0.2651		$a_{311} =$	0.2752	$a_{312} =$	0.2441
	$a_{411}^0 =$	0.2431		$a_{411} =$	0.2385	$a_{412} =$	0.2747
matrices	A_1^0			A_1		A_2	
	0.2417	0.2267		0.2361	0.2267	140	140
	0.25	0.2543		0.25	0.2543	130	110
	0.2651	0.2441		0.2752	0.2441	120	150
	0.2431	0.2747		0.2385	0.2747	130	120
sum	1	1		1	1	520	520

Table A4. *Cont.*

	Criteria					
normalized matrices	$\hat{\mathbf{A}}_1^0$		$\hat{\mathbf{A}}_1$		$\hat{\mathbf{A}}_2$	
	0.2417	0.2267	0.2361	0.2267	0.2692	0.2692
	0.25	0.2543	0.25	0.2543	0.25	0.2115
	0.2651	0.2441	0.2752	0.2441	0.2307	0.2884
	0.2431	0.2747	0.2385	0.2747	0.25	0.2307
weights	0.5	0.5	0.5	0.5	0.6	0.4
performances	$a_{11}^0 =$	0.2342	$a_{11} =$	0.2314	$a_{12} =$	0.2692
	$a_{21}^0 =$	0.2521	$a_{21} =$	0.2521	$a_{22} =$	0.2346
	$a_{31}^0 =$	0.2546	$a_{31} =$	0.2596	$a_{32} =$	0.2538
	$a_{41}^0 =$	0.2589	$a_{41} =$	0.2566	$a_{42} =$	0.2423
matrices	A^0		A			
	0.2342	0.2692	0.2314	0.2692		
	0.2521	0.2346	0.2521	0.2346		
	0.2546	0.2538	0.2596	0.2538		
	0.2589	0.2423	0.2566	0.2423		
sum	1	1	1	1		
normalized matrices	$\hat{A}^0$		$\hat{A}$			
	0.2342	0.2692	0.2314	0.2692		
	0.2521	0.2346	0.2521	0.2346		
	0.2546	0.2538	0.2596	0.2538		
	0.2589	0.2423	0.2566	0.2423		
weights	0.4	0.6	0.4	0.6		
performances	$a_1^0 =$	0.2552	$a_1 =$	0.2541		
	$a_2^0 =$	0.2416	$a_2 =$	0.2416		
	$a_3^0 =$	0.2541	$a_3 =$	0.2561		
	$a_4^0 =$	0.2489	$a_4 =$	0.2480		

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